ON THE SHARPNESS OF MEIER'S ANALOGUE OF FATOU'S THEOREM

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ABSTRACT

Meier's topological analogue of Fatou's theorem is shown to be sharp by exhibiting a bounded holomorphic function in the unit disk for which no point of a prescribed set of first category on the unit circle is a Meier point.

Let Γ be the unit circle and D be the open unit disk in the complex plane. We say that almost every point of Γ has a certain property, provided that the exceptional set is a subset of Γ of Lebesgue measure zero. Similarly, we say that nearly every point of Γ possesses a certain property, provided that the exceptional set is a subset of Γ of first Baire category.

The celebrated theorem of Fatou (see [5, p. 5]) asserts that if $f(z)$ is a bounded holomorphic function in D, then f has an angular limit at almost every point of Γ. Lusin and Priwaloff [3, pp. 156–159] (see also [2] and [6]) have shown that Fatou's theorem is sharp by proving that if E is a subset of Γ of measure zero, then there exists a bounded holomorphic function in D that has no asymptotic value at any point of E.

Meier [4, p. 330, Theorem 6] has recently obtained a topological analogue of Fatou's theorem that may be formulated as follows. If $f(z)$ is a function defined in D, and if $\zeta \in \Gamma$, then the cluster set of f at ζ is denoted by $C(f,\zeta)$ (the rudiments of cluster-set theory are to be found in [5]). The chordal principal cluster set of f at ζ is defined to be

$$\prod_{\chi} (f,\zeta) = \bigcap_{X} C_X(f,\zeta),$$

where X ranges over all chords (of the unit circle) at ζ and $C_X(f,\zeta)$ stands for the cluster set of f at ζ along X. We call a point $\zeta \in \Gamma$ a Meier point of f, provided that

$$\prod_{\chi}(f,\zeta) = C(f,\zeta) \subset \Omega, \tag{1}$$

where "$\subset$" symbolizes proper set inclusion and Ω represents the Riemann sphere. Meier's theorem asserts that if $f(z)$ is a bounded holomorphic function in D, then nearly every point of Γ is a Meier point of f.

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Before considering the question of the sharpness of this theorem, let us remark that Meier's theorem is not a trivial consequence of Fatou's. For if a point $\zeta \in \Gamma$ at which f has an angular limit is called a Fatou point of f, then there exists in D a bounded holomorphic function of which nearly every point of Γ is a Meier point but not a Fatou point. An example of such a function is a Blaschke product $b(z)$ with the property [1, p. 1070] that, for nearly every point $\zeta \in \Gamma$,

$$C_\rho(f,\zeta) = C(f,\zeta) = \Gamma \cup D \subset \Omega,$$

where $C_\rho(f,\zeta)$ is the radial cluster set of f at ζ.

Our aim is to prove the following

THEOREM. *Let E be a subset of Γ of first category. Then there exists a bounded univalent holomorphic function $f(z)$ in D such that no point of E is a Meier point of f.*

Proof. Since E is of first category, we may write

$$E = E_1 \cup E_2 \cup \cdots \cup E_n \cup \cdots,$$

where each E_n is a nowhere dense subset of Γ. Denote the closure of E_n by $\bar{E}_n$, and define

$$E' = \bar{E}_1 \cup \bar{E}_2 \cup \cdots \cup \bar{E}_n \cup \cdots.$$

Since each $\bar{E}_n$ is also nowhere dense, the set E' is an F_σ of first category. According to Lohwater and Piranian [2, p. 7, Theorem 1′], there exists a bounded univalent holomorphic function $f(z)$ in D with the property that at every point $\zeta \in \Gamma$, $f(z)$ has a radial limit, call it $f(\zeta)$, and $f(\zeta)$, regarded as a function of ζ along Γ, is discontinuous at every point of E' (and is continuous at every point of $\Gamma - E'$). Now it is seen directly that if $f(z)$ $(z \in D)$ has a global limit at a point $\zeta \in \Gamma$, then ζ is a point of continuity of the function $f(\zeta)$ $(\zeta \in \Gamma)$. Consequently $f(z)$ does not have a global limit at any point $\zeta \in E'$. (On the other hand, according to Weniaminoff [7, p. 92, Lemma 2], $f(z)$ has the global limit $f(\zeta)$ at every point $\zeta \in \Gamma - E'$). This means that for every $\zeta \in E'$ we have

$$C_\rho(f,\zeta) = \{f(\zeta)\} \subset C(f,\zeta),$$

and hence

$$\prod_\chi(f,\zeta) \subset C(f,\zeta),$$

so that (1) is not satisfied, and therefore ζ is not a Meier point of f. Since $E \subseteq E'$, the proof of the theorem is complete.

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